ECON 441

Introduction to Mathematical Economics

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Final Exam Review

Numbers, Sets, and Functions

- Types of numbers: integers, fractions, rational numbers, irrational numbers, real numbers.
- Set notation:
  Example: $A = \{a, b, c, d\}$ *or* $A = \{x | x \in \mathbb{R}\}$
- Set relations: equivalence, subset, disjoint
- Set operations: union, intersection, complement

Numbers, Sets, and Functions

- Cartesian product
  Example: $\mathbb{R}^2 = \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$
- Relation: subset of a Cartesian product
- Function: a relation where for each x there is a unique y

$$f : X \to Y, \quad y = f(x)$$

  X : domain, Y : codomain, $f(X)$: range

Numbers, Sets, and Functions

- One-to-one function: each value of y is also associated with a unique value of x
- One-to-one mapping unique to strictly monotonic functions
- Inverse of a function only exists for strictly monotonic functions

  $$x = f^{-1}(y)$$

  returns the value corresponding value of x for each y.

Summation Notation

Example 1:

$$\sum_{i=1}^{3} \sum_{j \leqslant i} X_i Y_j$$

Summation Notation

Example 2:

$$\sum_{i=1}^{2}\sum_{j=1}^{2}(X_i Y_j + 4Y_j^2 + 1)$$

Linear Algebra

- Matrix operations: addition, subtraction, scalar multiplication, matrix multiplication
- Identity matrix, transpose of a matrix
- Inverse of a matrix: $AA^{-1} = A^{-1}A = I$
- Solution of a linear-equation system $Ax = b$

$$A^{-1}Ax = A^{-1}b \rightarrow x = A^{-1}b$$

- Finding the determinant $|A|$ and inverse of a matrix

$$A^{-1} = \frac{1}{|A|}AdjA$$

Linear Algebra

- If a matrix's inverse exists, it's called a **nonsingular** matrix
- Necessary and sufficient conditions for nonsingularity:
  - *Necessary*: square matrix
  - *Sufficient*: rows (or equivalently columns) are linearly independent
- Rank of matrix: maximum number of linearly independent rows (square matrix with full rank = nonsingular)
- For singular matrices the determinant $|A| = 0$

Linear Algebra

Say we have the following system of equations:

$$3x + 2y = 20$$
$$6x + 4y = 40$$

Can write this as:

$$Av = b$$

where

$$A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \quad v = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} 20 \\ 40 \end{bmatrix}$$

Unique solution for this system does not exist as A is singular.

Linear Algebra

Let's solve the following system of equations

$$
\begin{aligned}
3x + 2y &= 20 \\
6x - 3y &= 40
\end{aligned}
$$

Calculus

- Limit definition of differentiability and continuity
- Rules of differentiation to differentiate functions (including log and exponential functions)
- Partial and total derivatives
- Second-order derivatives
- Elasticities and partial elasticities

Calculus

For the function:

$$y = f(x_1, x_2, ..., x_n)$$

Note that the gradient and Hessian is given by

$$\nabla f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \qquad H = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{bmatrix}$$

Calculus

Calculate the total differential of the production function

$$Q = F(K, L)$$

to find how a small change in both labor and capital affects the production.

Calculus

Consider a company that allocates its marketing budget $X(a)$ based on the economic climate, represented by an economic index a.

$$X(a) = 10a$$

The company's sales revenue Y depends on both the marketing spend $X(a)$ and the economic index a.

$$Y(X(a), a) = X(a) \cdot \log(1 + a)$$

How does this company's revenue vary with respect to the economic index a?

Single-Variable Optimization

- Given a function
$$y = f(x)$$
- Critical point $f'(x^*) = 0$, necessary condition for an optimum
- Sufficient condition:
  - maximum if $f''(x^*) < 0$
  - minimum if $f''(x^*) > 0$

Single-Variable Optimization

Let's find the extrema for the following function and plot it:

$$f(x) = x^4 - 2x^2$$

Single-Variable Optimization

Say, $f(x)$ is a strictly concave function and

$$f'(x^*) = 0$$

Is $f(x^*)$ the global maximum? Can you explain why?

Multiple-Variable Optimization

$$y = f(x_1, x_2, ..., x_n)$$

First-order condition:

$$\nabla f(x_1, x_2, ..., x_n) = 0$$

That is:

$$\begin{aligned} f_1(x_1, x_2, ..., x_n) &= 0 \\ f_2(x_1, x_2, ..., x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, ..., x_n) &= 0 \end{aligned}$$

Multiple-Variable Optimization

$$\pi(K, L) = AK^{1/3}L^{2/3} - wL - rK$$

Show that at the optimal: $wL = 2rK$

Envelope Theorem

Value function:

$$V = f\left(x^*(\alpha), y^*(\alpha), \alpha\right)$$

If we differentiate V with respect to α:

$$\frac{dV}{d\alpha} = f_x^* \cdot \frac{dx^*}{d\alpha} + f_y^* \cdot \frac{dy^*}{d\alpha} + f_\alpha^*$$

From the first order conditions we know $f_x^* = f_y^* = 0$, therefore

$$\frac{dV}{d\alpha} = f_\alpha^*$$

Envelope Theorem

$$V = \pi(K^*, L^*)$$

How does optimal profit change due to a change in w or r?

Constrained Optimization

$$U = U(c_1, c_2) = \ln c_1 + \beta \ln c_2 \qquad 0 < \beta < 1$$

- $y_1, y_2 > 0$: income in period 1 and 2
- Income you save s in period 1 earns interest $r > 0$
- In which case,

$$c_1 + s = y_1 \qquad c_2 = y_2 + (1 + r)s$$

- Combining these constraints:

$$c_1 + \frac{1}{1+r} c_2 = \underbrace{y_1 + \frac{1}{1+r} y_2}_{m \equiv \text{present-discounted income}}$$

Constrained Optimization

$$\max_{\{c_1, c_2\}} \quad U(c_1, c_2) = \ln c_1 + \beta \ln c_2 \quad s.t. \quad c_1 + \frac{1}{1+r} c_2 = m$$

Envelope Theorem with Constraints

Value function:

$$V = f\left(x^*(\alpha), y^*(\alpha), \alpha\right)$$

By envelope theorem:

$$\frac{dV}{d\alpha} = \frac{\partial L^*}{\partial \alpha}$$

How does the optimal utility change due to changes in r, y_1, y_2, or β?

Interpretation of the Lagrange Multiplier

Lagrangian function:

$$L = f(x, y) + \lambda[c - g(x, y)]$$

Substituting the solutions into the objective function, we get

$$V = f(x^*(c), y^*(c))$$

By the envelope theorem,

$$\frac{dV}{dc} = \frac{\partial L^*}{\partial c} = \lambda^*$$

Global Optimizers with Constraints

Consider the problem:

Maximize $f(x_1, x_2, ..., x_n)$ subject to $g(x_1, x_2, ..., x_n) = k$.

The stationary point $(x_1^*, x_2^*, ..., x_n^*)$ of the lagrangian is a global maximum if:

1. $f(x_1, x_2, ..., x_n)$ is quasiconcave
2. The constraint set is convex

A Few Last Words

Please fill the SOQs :)

Thanks for a great semester.

Good luck and don't be a stranger!