Homework 10 Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

<u>Exercise 12.2</u>

1. (a) $z = xy$ s.t. $x + 2y = 2$

   Setting up the Lagrangian:

$$L(x, y, \lambda) = xy + \lambda(2 - x - 2y)$$

   First-order conditions (F.O.C.s):

$$\frac{dL}{dx} = y - \lambda = 0 \quad (1)$$
$$\frac{dL}{dy} = x - 2\lambda = 0 \quad (2)$$
$$\frac{dL}{d\lambda} = 2 - x - 2y = 0 \quad (3)$$

   From (1) and (2), $y = \lambda$ and $x = 2\lambda$, so

$$\frac{y}{x} = \frac{\lambda}{2\lambda} \rightarrow \quad x = 2y$$

   Pugging in $x = 2y$ in (3):

$$2 - x - x = 0 \rightarrow x = 1$$

   Stationary point: $x^* = 1, y^* = 1/2$. Also note that from (1), $\lambda^* = y^* = 1/2$.

   (b) $2 = x(y + 4)$ s.t. $x + y = 8$

   Lagrangian function:

$$L(x, y, \lambda) = x(y + 4) + \lambda(8 - x - y)$$

First-order conditions (F.O.C.s):

$$L_x = y + 4 - \lambda = 0 \tag{1}$$
$$L_y = x - \lambda = 0 \tag{2}$$
$$L_\lambda = 8 - x - y = 0 \tag{3}$$

From equations (1) and (2):

$$\frac{y+4}{x} = \frac{\lambda}{\lambda} \rightarrow y + 4 = x$$

Plugging $x = y + 4$ in equation (3):

$$8 - y - 4 - y = 0 \rightarrow y^* = 2$$

Stationary point: $x^* = y^* + 4 = 6, y^* = 2$. Also note, $\lambda^* = x^* = 6$.

(c) $f(x, y) = x - 3y - xy$ s.t. $x + y = 6$

Lagrangian function :

$$L(x, y, \lambda) = x - 3y - xy + \lambda(6 - x - y)$$

First-order conditions (F.O.C.s):

$$\frac{dL}{dx} = 1 - y - \lambda = 0 \tag{1}$$
$$\frac{dL}{dy} = -3 - x - \lambda = 0 \tag{2}$$
$$\frac{dL}{d\lambda} = 6 - x - y = 0 \tag{3}$$

From equations (1) and (2):

$$1 - y = -3 - x \rightarrow y = x + 4$$

Plugging in (3):

$$6 - x - x - 4 = 2 - 2x = 0 \rightarrow x = 1$$

Stationary point: $x^* = 1, y^* = 5$
Lagrange multiplier, $\lambda^* = 1 - y^* = -4$

(d) $z = 7 - y + x^2$ s.t. $x + y = 0$

Lagrangian function:

$$L(x, y, \lambda) = 7 - y + x^2 + \lambda(-x - y)$$

First-order conditions (F.O.C.s):

$$L_x = 2x - \lambda = 0 \quad (1)$$
$$L_y = -1 - \lambda = 0 \quad (2)$$
$$L_\lambda = -x - y = 0 \quad (3)$$

From equation (2), $\lambda^* = -1$ From equation (1), $x^* = \frac{\lambda^*}{2} = -\frac{1}{2}$ From equation (3), $y^* = -x^* = \frac{1}{2}$
So the Stationary point: $(x^*, y^*) = \left(\frac{-1}{2}, \frac{1}{2}\right)$

2. Suppose we are interested in finding the optimal value of $f(x, y)$ subject to the constraint $g(x, y) = c$. We would start by setting up the Lagrangian function:

$$L(x, y, \lambda) = f(x, y) + \lambda(c - g(x, y))$$

At the optimizing point (x^*, y^*), the following first-order conditions hold:

$$L_x(x^*, y^*, \lambda^*) = f_x(x^*, y^*) - \lambda^* g_x(x^*, y^*) = 0 \quad (1)$$
$$L_y(x^*, y^*, \lambda^*) = f_y(x^*, y^*) - \lambda^* g_y(x^*, y^*) = 0 \quad (2)$$
$$L_\lambda(x^*, y^*, \lambda^*) = c - g(x^*, y^*) = 0 \quad (3)$$

Now note that implicitly the optimal inputs x^* and y^* depend on c. So we could express x^* and y^* as functions of c, i.e., $x^*(c)$ and $y^*(c)$. Then optimal value of f is given by $f(x^*(c), y^*(c))$.
Now, say, we want to know how the optimal value of f changes if we relax

the constraint i.e. if we increase c slightly. To find this we can differentiate $f(x^*(c), y^*(c))$ with respect to c, then by chain-rule:

$$\frac{d}{dc} f(x^*(c), y^*(c)) = f_x(x^*, y^*) \cdot \frac{dx^*}{dc} + f_y(x^*, y^*) \frac{dy^*}{dc} \tag{4}$$

Now note that from (1) and (2), we have $f_x(x^*, y^*) = \lambda^* g_x(x^*, y^*)$ and $f_y(x^*, y^*) = \lambda^* g_y(x^*, y^*)$. Plugging these terms in equation (4), we get:

$$\frac{d}{dc} f(x^*(c), y^*(c)) = \lambda^* \underbrace{\left[g_x(x^*, y^*) \cdot \frac{dx^*}{dc} + g_y(x^*, y^*) \frac{dy^*}{dc} \right]}_{=1}$$

The term in the parenthesis is 1 because if we take the derivative of (3) with respect to c, we get

$$g_x(x^*, y^*) \cdot \frac{dx^*}{dc} + g_y(x^*, y^*) \frac{dy^*}{dc} = 1$$

So we have that,

$$\frac{d}{dc} f(x^*(c), y^*(c)) = \lambda^*$$

In which case, λ^* tells us what happens to the optimal value of the function by relaxing the constraint. Whenever, $\lambda^* > 0$, the optimal value increases and whenever $\lambda^* < 0$, it decreases.

Note: In the class, we came to the above conclusion by taking the derivative of L with respect to c. We showed that $dL(x^*, y^*, \lambda^*)/dc = \lambda^*$. Both approaches are equivalent because at the optimal value $L = f$ as the constraint always binds. In the hindsight, I think the proof I outline here is slightly more intuitive.

3. (a) $L(x, y, \omega, \lambda) = x + 2y + 3\omega + xy - y\omega + \lambda(10 - x - y - 2\omega)$

First-order conditions:

$$
\begin{aligned}
L_x &= 1 + y - \lambda = 0 \\
L_y &= 2 + x - \omega - \lambda = 0 \\
L_\omega &= 3 - y - 2\lambda = 0 \\
L_\lambda &= 10 - x - y - 2\omega = 0
\end{aligned}
$$

(b) $L(x, y, \omega, \lambda) = x^2 + 2xy + y\omega^2 + \lambda\left(24 - 2x - y - \omega^2\right) + \mu(8 - x - \omega)$

First-order conditions:

$$
\begin{aligned}
L_x &= 2x + 2y - 2\lambda - \mu = 0 \\
L_y &= 2x + \omega^2 - \lambda = 0 \\
L_\omega &= 2y\omega - 2\omega\lambda - \mu = 0 \\
L_\lambda &= 24 - 2x - y - \omega^2 = 0 \\
L_\mu &= 8 - x - \omega = 0
\end{aligned}
$$

4. $L(x, y, \lambda) = f(x, y) + \lambda(-G(x, y))$

First-order conditions:

$$
\begin{aligned}
L_x &= f_x - \lambda G_x = 0 \\
L_y &= f_y - \lambda G_y = 0 \\
L_\lambda &= -G(x, y) = 0
\end{aligned}
$$