# ECON 441

## Introduction to Mathematical Economics

Div Bhagia

Lecture 11

Envelope Theorem, Quasiconcavity,
Convex sets, Homogenous Functions

# Profit Maximization

Maximize:

$$\pi(L) = AL^{\alpha} - wL, \qquad 0 < \alpha < 1$$

First-order condition:

$$\pi'(L) = \alpha AL^{\alpha-1} - w = 0 \rightarrow L^* = \left(\frac{\alpha A}{w}\right)^{\frac{1}{1-\alpha}}$$

Optimal labor input is a function of wages, we can write $L^*(w)$.

Also, we wrote $\pi()$ as a function of $L$, but it also depends on $w$ (and $\alpha$) which are *exogenous* parameters.

# Profit Maximization

We are often interested in questions like:

> How does the profit change due to change in wages?

Remember,

$$\pi(L) = AL^{\alpha} - wL$$

So optimal profit depends directly on $w$ but also indirectly through $L$

# Maximum Value Function

Maximum value function: objective function after plugging in optimal values for the choice *variables*

Maximum value function is a function of *parameters*

For the profit maximization problem, the value function:

$$V(w) = \pi(L^*(w), w) = AL^*(w)^\alpha - wL^*(w)$$

Plugging in $L^*(w) = \left(\frac{A\alpha}{w}\right)^{\frac{1}{1-\alpha}}$, we get:

$$V(w) = A\left(\frac{\alpha A}{w}\right)^{\frac{\alpha}{1-\alpha}} - w\left(\frac{\alpha A}{w}\right)^{\frac{1}{1-\alpha}}$$

# Profit Maximization

Value function:

$$V(w) = \pi^* = \pi(L^*(w), w)$$

To see how optimal profit $V(w)$ changes with wages:

$$V'(w) = \underbrace{\pi_L^* \cdot \frac{dL^*}{dw}}_{\textit{Indirect Effect}} + \underbrace{\pi_w^*}_{\textit{Direct Effect}}$$

# Profit Maximization

Value function:

$$V(w) = \pi^* = \pi(L^*(w), w)$$

To see how optimal profit $V(w)$ changes with wages:

$$V'(w) = \underbrace{\pi_L^* \cdot \frac{dL^*}{dw}}_{\textit{Indirect Effect}} + \underbrace{\pi_w^*}_{\textit{Direct Effect}}$$

But by the F.O.C, $\pi_L^* = 0$, so

$$V'(w) = \pi_w^*$$

# Envelope Theorem

$$V'(w) = \pi^*_w$$

This result says that at the optimum, as wages vary, with labor allowed to adjust optimally gives the same result as if labor was held fixed.

In other words, only the direct effect of wages matters, even though it enters indirectly through choice of labor input as well.

This is actually a general result called the *envelope theorem*.

# Profit Maximization

How does the profit change due to a change in wages?

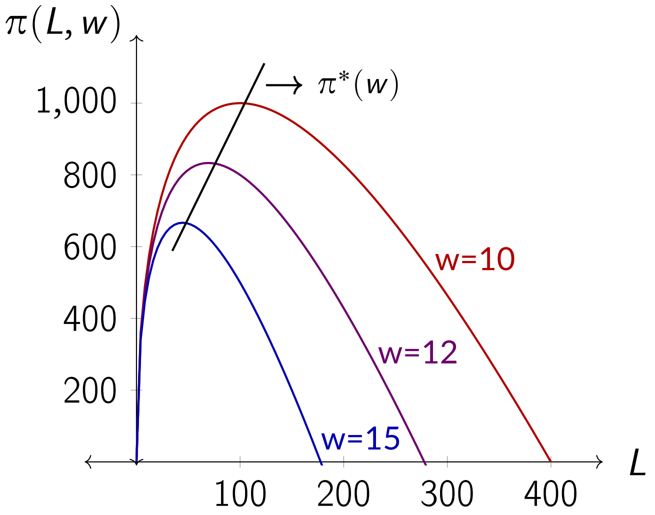
$$\pi^* = AL^*(w)^\alpha - wL^*(w)$$

The answer is given by:

$$\pi^*_w = -L^*(w) = -\left(\frac{\alpha A}{w}\right)^{\frac{1}{1-\alpha}}$$

Note that you would get the same answer by calculating $V'(w)$ using the expression on slide 3, but the envelope theorem suggests the above shortcut.

# Envelope Theorem



# Envelope Theorem

More generally, consider the following maximization problem with two choice variables $x$ and $y$, and one parameter, $\alpha$:

Maximize

$$U = f(x, y, \alpha)$$

The first order necessary conditions are

$$f_x(x, y, \alpha) = f_y(x, y, \alpha) = 0$$

If the second-order conditions are met, these two equations implicitly define the solutions $x = x^*(\alpha) \quad y = x^*(\alpha)$.

# Envelope Theorem

If we substitute these solutions into the objective function, we obtain a new function:

$$V(\alpha) = f\left(x^*(\alpha), y^*(\alpha), \alpha\right)$$

If we differentiate $V$ with respect to $\alpha$:

$$\frac{dV}{d\alpha} = f_x \frac{\partial x^*}{\partial \alpha} + f_y \frac{\partial y^*}{\partial \alpha} + f_\alpha$$

From the first order conditions we know $f_x = f_y = 0$, therefore

$$\frac{dV}{d\alpha} = f_\alpha$$

# Envelope Theorem with Constraints

We have a similar envelope theorem for constrained optimization problems such as

Maximize $U = f(x, y; \alpha)$ subject to $G(x, y; \alpha) = 0$

Lagrangian function:

$$L = f(x, y; \alpha) + \lambda G(x, y; \alpha)$$

# Envelope Theorem with Constraints

Lagrangian function:

$$L = f(x, y; \alpha) + \lambda G(x, y; \alpha)$$

First-order conditions:

$$\begin{aligned} L_x &= f_x + \lambda G_x = 0 \\ L_y &= f_y + \lambda G_y = 0 \\ L_\lambda &= G(x, y; \alpha) = 0 \end{aligned}$$

Solving this system of equations gives us

$$x = x^*(\alpha) \quad y = y^*(\alpha) \quad \lambda = \lambda^*(\alpha)$$

# Envelope Theorem with Constraints

Substituting the solutions into the objective function, we get

$$U^* = f\left(x^*(\alpha), y^*(\alpha), \alpha\right) = V(\alpha)$$

By envelope theorem:

$$\frac{dV(\alpha)}{d\alpha} = \frac{\partial L^*}{\partial \alpha}$$

# Interpretation of the Lagrange Multiplier

An application of the envelope theorem for constrained optimization gives us the interpretation of the Lagrange multiplier.

Consider the following problem:

Maximize $U = f(x, y)$ subject to $g(x, y) = c$

Can think of the constraint as:

$$G(x, y, c) = c - g(x, y)$$

So $c$ is just a parameter.

# Interpretation of the Lagrange Multiplier

Lagrangian function:

$$L = f(x, y) + \lambda[c - g(x, y)]$$

Substituting the solutions into the objective function, we get

$$U^* = V(c) = f(x^*(c), y^*(c))$$

By the envelope theorem,

$$\frac{dV(c)}{dc} = \frac{\partial L^*}{\partial c} = \lambda^*$$

# The Quiz Question

# Should you get married?

- Two agents $a$ and $b$ with incomes $y_a$ and $y_b$, respectively
- Let $q_a$ and $q_b$ denote consumption of agent $a$ and $b$, respectively
- $Q$ denotes consumption of public good
- If single, agent $s$ maximizes

$$U_s(Q, q_s) \quad \text{subject to } Q + q_s = y_s$$

- Budget constraint when married:

$$Q + q_a + q_b = y_a + y_b$$

# Concave and Convex Functions

- Concave function: $f''(x) \leqslant 0$ for all $x$
- Convex function: $f''(x) \geqslant 0$ for all $x$

- Strictly concave function: $f''(x) < 0$ for all $x$
- Strictly convex function: $f''(x) > 0$ for all $x$

# Global Optimizers

If a function is concave, any critical point will give us an absolute maximum.

If a function is strictly concave, any critical point will give us a *unique* absolute maximum.

If a function is convex, any critical point will give us an absolute minimum.

If a function is strictly convex, any critical point will give us a *unique* absolute minimum.

# Concave and Convex Functions

$f$ is concave if:

$$f(\lambda x_1 + (1-\lambda)x_2) \geqslant \lambda f(x_1) + (1-\lambda)f(x_2)$$

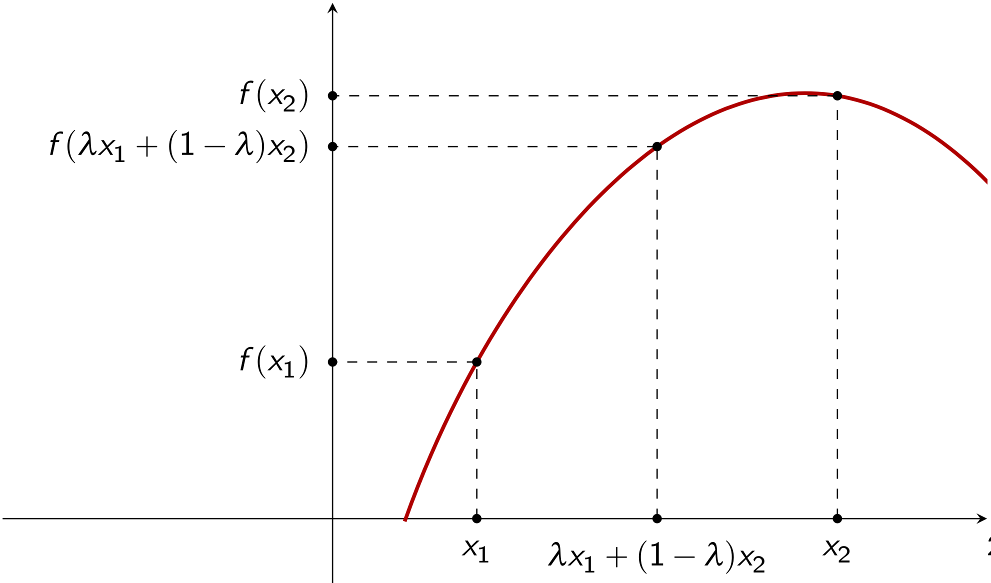
$f$ is convex if:

$$f(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda f(x_1) + (1-\lambda)f(x_2)$$

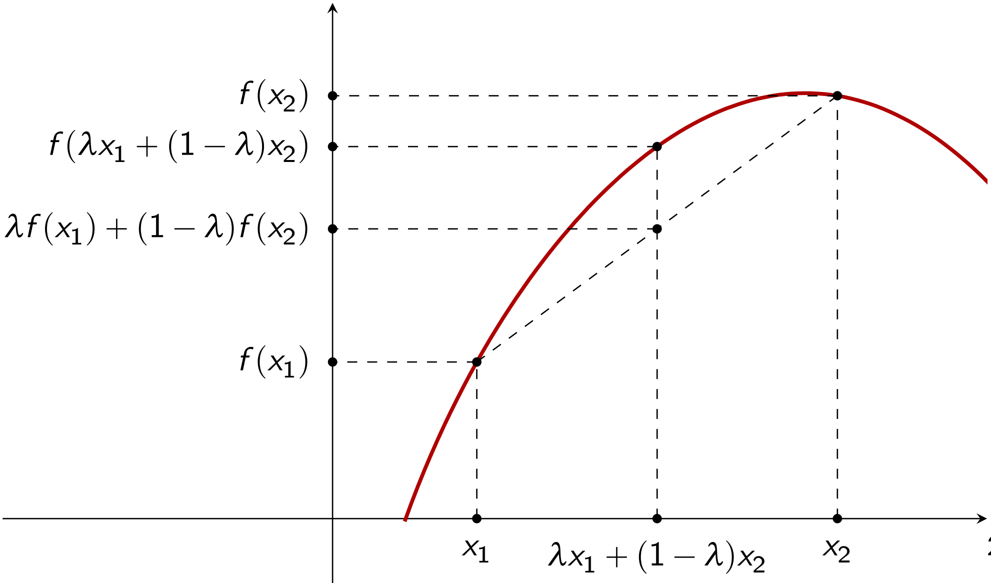
where $\lambda \in (0,1)$.

For strict concavity/convexity replace with strict inequalities.

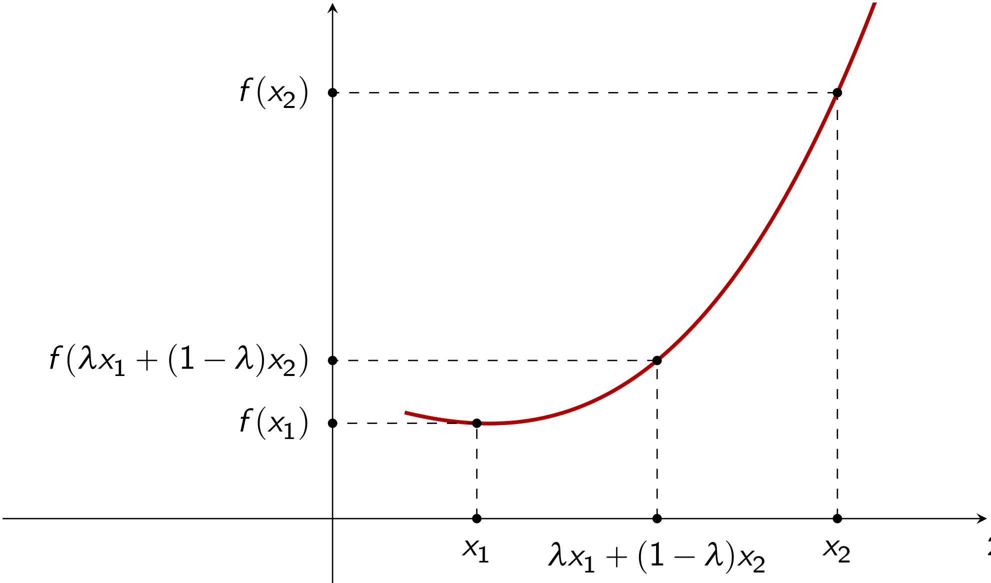
# Concave Function



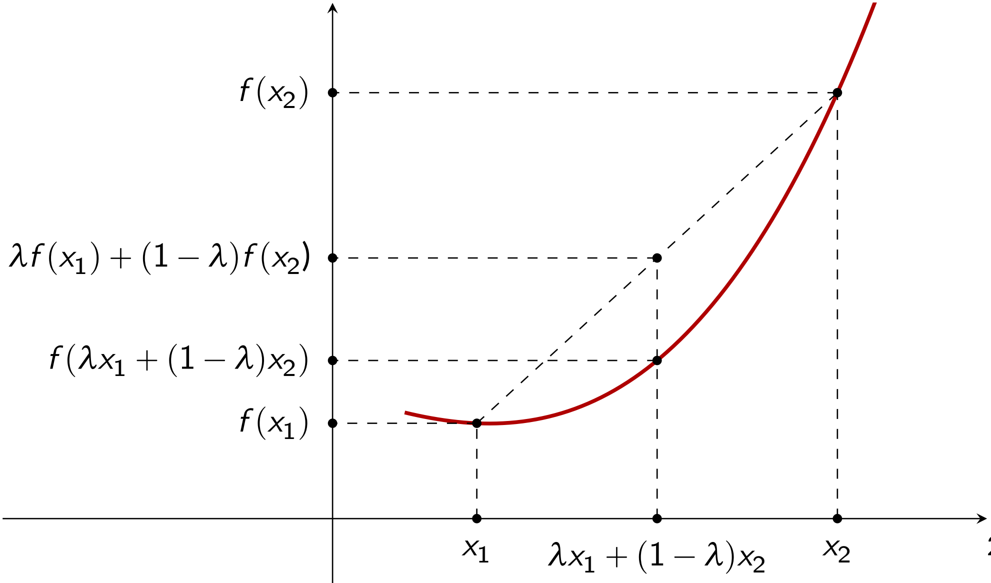
# Concave Function



# Convex Function



# Convex Function



# Concavity and Convexity

Can extend the concept of concavity and convexity to multi-variable functions.

A function is concave iff, for any distinct points $u$ and $v$ and any $0 < \lambda < 1$,

$$\lambda f(u) + (1 - \lambda) f(v) \leqslant f(\lambda u + (1 - \lambda) v)$$

Similarly, a function is convex iff

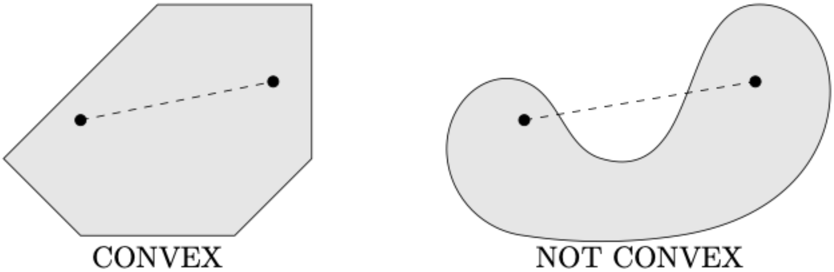
$$\lambda f(u) + (1 - \lambda) f(v) \geqslant f(\lambda u + (1 - \lambda) v)$$

Substituting strict inequalities in the above, we get the definitions of strict concavity and convexity.

# Convex sets

Convex set different from convex function.

A set $A$ is *convex* if for any $x, y \in A$, $(1-\lambda)x + \lambda y$ also belongs to $A$ where $\lambda \in [0,1]$.



# Quasiconcavity and Quasiconvexity

A function is *quasiconcave* if and only if for any pair of distinct points $u$ and $v$ in the convex domain of $f$, and for $0 < \lambda < 1$, we have

$$f(\lambda u + (1-\lambda)v) \geqslant \min\{f(u), f(v)\}$$

Note that when $f(v) \geqslant f(u)$, the above inequality becomes

$$f(\lambda u + (1-\lambda)v) \geqslant f(u)$$

Replace inequality with strict inequality to get the definition of strict quasiconcavity.

# Quasiconcavity and Quasiconvexity

A function is *quasiconvex* if and only if for any pair of distinct points $u$ and $v$ in the convex domain of $f$, and for $0 < \lambda < 1$, we have

$$f(\lambda u + (1-\lambda)v) \leqslant \max\{f(u), f(v)\}$$

Note that when $f(v) \geqslant f(u)$, the above inequality becomes

$$f(\lambda u + (1-\lambda)v) \leqslant f(v)$$

Replace inequality with strict inequality to get the definition of strict quasiconvexity.

# Quasiconcavity and Quasiconvexity

- If $f(x)$ is (strictly) quasiconcave, then $-f(x)$ is (strictly) quasiconvex.
- Any (strictly) concave (convex) function is (strictly) quasiconcave (quasiconvex), but the converse may not be true.
- If $f(x)$ is linear, then it is quasiconcave as well as quasiconvex.

# Alternative Definitions

A function $f(x)$, where $x$ is a vector of variables is *quasiconcave* iff for any constant $k$, the upper-contour set

$$S^U = \{x | f(x) \geqslant k\}$$

is a convex set.

A function $f(x)$, where $x$ is a vector of variables is *quasiconvex* iff for any constant $k$, the lower-contour set

$$S^L = \{x | f(x) \leqslant k\}$$

is a convex set.

# Global Optimizers with Constraints

Consider the problem:

Maximize $f(x_1, x_2, ..., x_n)$ subject to $g(x_1, x_2, ..., x_n) = k$.

The stationary point $(x_1^*, x_2^*, ..., x_n^*)$ of the lagrangian is a global maximum if:

1. $f(x_1, x_2, ..., x_n)$ is quasiconcave
2. The constraint set is convex

# Homogeneous Functions

A function is said to be homogeneous of degree $k$ if

$$f(ax_1, ax_2, \ldots, ax_n) = a^k f(x_1, x_2, \ldots x_n)$$

Example: $f(x_1, x_2) = x_1 + x_2$ is homogenous of degree 1.

*Find the degree of homogeneity for* $f(x, y) = x^2 + xy$.

What about $f(x, y) = x^2 + y$?

# Example

$$f(K, L) = AK^{\alpha}L^{\beta}$$

# Homework and References

- Sections: 13.5, 11.5, 12.4, 12.6
- Homework
    - Exercise 11.5: 1 (a), 2(c), 4, 5
    - Exercise 12.4: 1, 2, 4
    - Exercise 12.6: 1, 2, 6, 7