# Homework 11 Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

*Exercise 11.5*

1. (a) $y = x^2$

   Take two distinct points $x_1$ and $x_2$ and $0 < \lambda < 1$, then

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &= (\lambda x_1 + (1-\lambda)x_2)^2 \\ &= \lambda^2 x_1^2 + (1-\lambda)^2 x_2^2 + 2\lambda(1-\lambda)x_1 x_2 \end{aligned} \tag{1}$$

   Also note that,

$$\lambda f(x_1) + (1-\lambda) f(x_2) = \lambda x_1^2 + (1-\lambda)x_2^2 \tag{2}$$

   Subtracting (2) from (1)

$$\begin{aligned} (1)-(2) &= \lambda^2 x_1^2 + (1-\lambda)^2 x_2^2 + 2\lambda(1-\lambda)x_1x_2 - \lambda x_1^2 - (1-\lambda)x_2^2 \\ &= \lambda(\lambda-1)x_1^2 - (1-\lambda)\lambda x_2^2 + 2\lambda(1-\lambda)x_1x_2 \\ &= \lambda(\lambda-1)\left(x_1^2 + x_2^2 - 2x_1x_2\right) \\ &= \lambda(\lambda-1)(x_1+x_2)^2 < 0 \end{aligned}$$

   Since $(1)-(2) < 0$,

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2)$$

   So $f$ is strictly convex.

2. (c) $f(x, y) = xy$

Take two distinct points $u$ and $v$ and $0 < \lambda < 1$, then

$$\begin{aligned} f(\lambda u + (1-\lambda)v) &= f\left(\lambda u_1 + (1-\lambda)v_1, \lambda u_2 + (1-\lambda)v_2\right) \\ &= \left(\lambda u_1 + (1-\lambda)v_1\right)\left(\lambda u_2 + (1-\lambda)v_2\right) \\ &= \lambda^2 u_1 u_2 + \lambda(1-\lambda)v_1 u_2 + \lambda(1-\lambda)u_1 v_2 + (1-\lambda)^2 v_1 v_2 \end{aligned} \tag{3}$$

Also note that,

$$\lambda f(u) + (1-\lambda)f(v) = \lambda u_1 u_2 + (1-\lambda)v_1 v_2 \tag{4}$$

Subtracting (4) from (3)

$$\begin{aligned} (4) - (3) =& \lambda(\lambda-1)u_1 u_2 + \lambda(1-\lambda)v_1 u_2 + \lambda(1-\lambda)u_1 v_2 - (1-\lambda)\lambda v_1 v_2 \\ =& \lambda(\lambda-1)\left[u_1 u_2 - v_1 u_2 - u_1 v_2 + v_1 v_2\right] \\ =& \lambda(\lambda-1)\left(\left(u_1 - v_1\right)u_2 - \left(u_1 - v_1\right)v_2\right)\right] \\ =& \lambda(\lambda-1)\left(u_1 - v_1\right)\left(u_2 - v_2\right) \end{aligned}$$

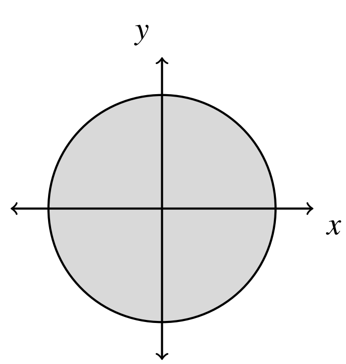
Since $(1) - (2) < 0$,

$$f\left(\lambda x_1 + (1-\lambda)x_2\right) < \lambda f\left(x_1\right) + (1-\lambda)f\left(x_2\right)$$

$f(.)$ is neither concave nor convex as (1) ≥ (2) sometimes and $(1) \leq (2)$ other times.

4. (a) No (b) No (c) Yes

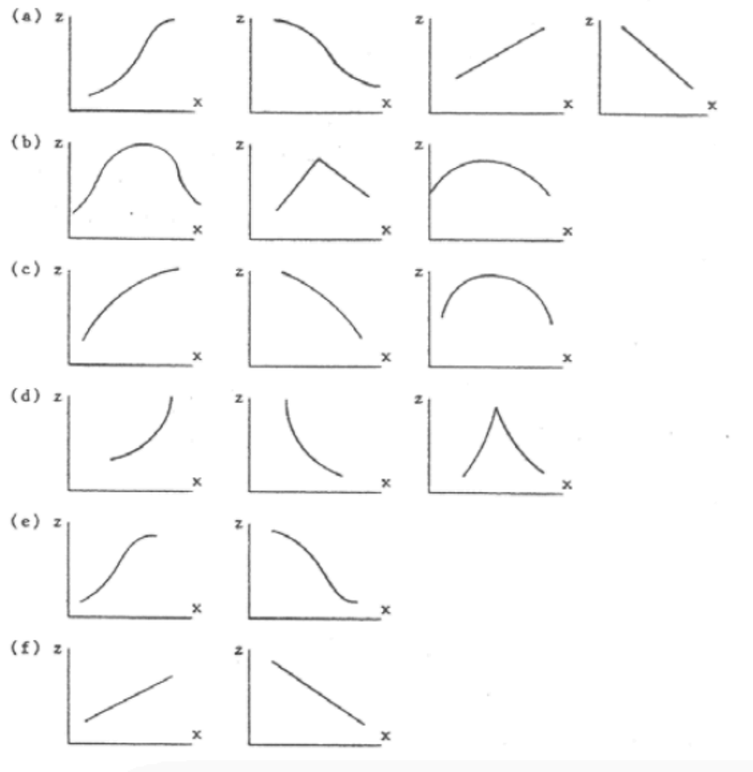
5. (a)




(b) Yes, convex.

*Exercise 12.4*

1. Examples of acceptable curves:

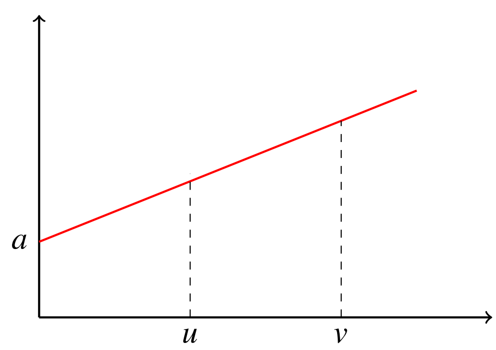



2. (a) $f(x) = a$

   Quasiconcave but not strictly so because for $u, v$ s.t $f(u) \geqslant f(v)$:

$$f(\lambda u + (1-\lambda)v) = f(v) = a$$

   (b) $f(x) = a + bx \quad (b > 0)$

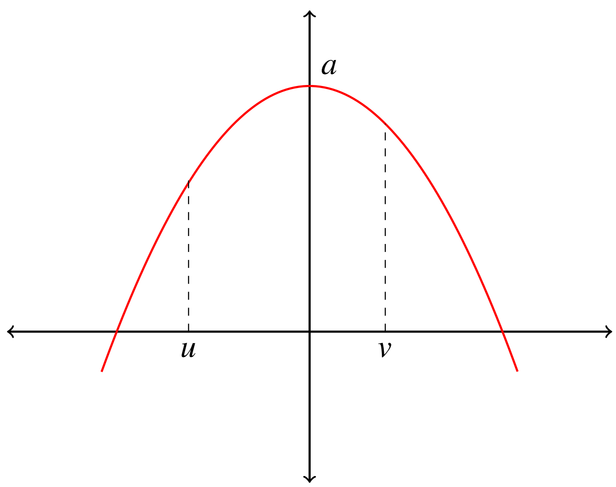



   For any point between $u$ and $v$ given by $\lambda u+(1-\lambda)v$, the value of the function $f(\lambda u + (1-\lambda)v)$ will be strictly greater than $f(u)$ as $f$ is a strictly increasing function. So $f(.)$ is strictly quasiconcave.

(c) $f(x) = a + cx^2 \quad (c < 0)$
To draw this function, let's calculate the first and the second derivatives:

$$f'(x) = 2cx, \qquad f''(x) = 2c < 0$$

Note that, for $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$. Moreover, at $f(0) = a$.

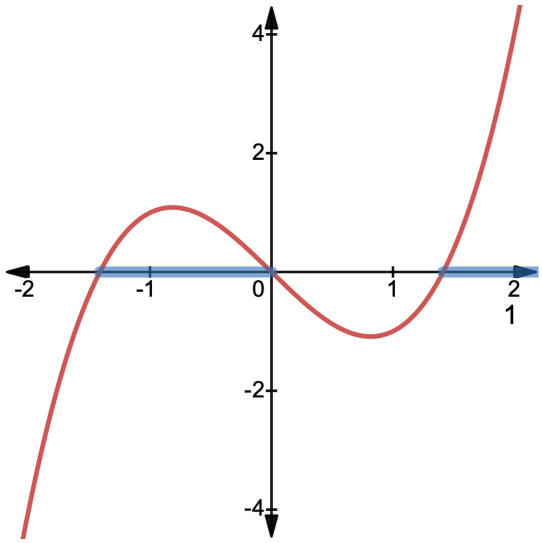



From the graph of the function, we can see that this function is strictly quasiconcave.

4. (a) $f(x) = x^3 - 2x$
In the graph below, the blue line highlights the following upper-contour set:

$$S^U = \{x | f(x) \geq 0\}$$

We can see from the graph that this is not a convex set. So this function is not quasiconcave. Similarly, the lower-contour set for this function is not convex as well and this function is not quasiconvex.




(b) $f(x_1, x_2) = 6x_1 - 9x_2$

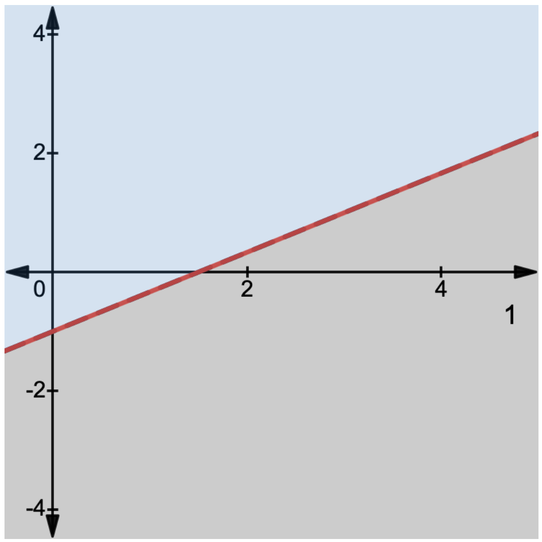
Note that the upper-contour set for this function at 0:

$$S^U = \{(x_1, x_2) | 6x_1 - 9x_2 \geq k\}$$

Note that, $6x_1 - 9x_2 = k \rightarrow x_2 = \frac{6x_1 - k}{9}$. So we can write the upper-contour set as:

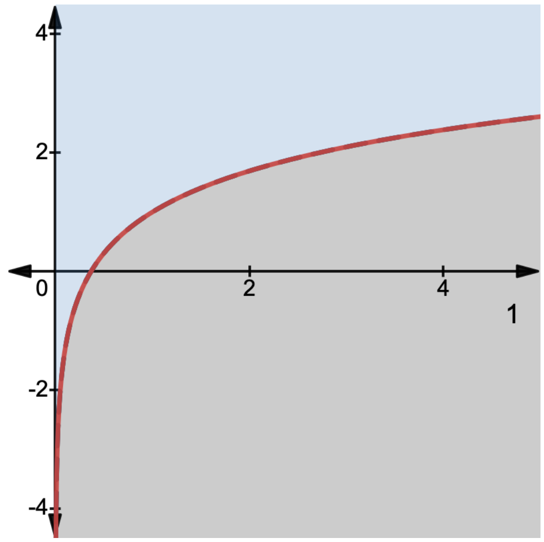
$$S^U = \left\{(x_1, x_2) | x_2 \leq \frac{6x_1 - k}{9}\right\}$$

This set is presented below and is convex. Hence, the function is quasiconcave. The lower-contour set is also convex and the function is quasiconvex as well. (Grey is the upper-contour set and blue is the lower-contour set.)



(c) $f(x_1, x_2) = x_2 - \ln x_1$

By similar reasoning as (b), this function is strictly quasiconcave but not quasiconvex. (Grey is the upper-contour set and blue is lower-contour set.)




*Exercise 12.6*

1. (a) $f(x, y) = \sqrt{xy}$

$$f(ax, ay) = \sqrt{(ax)(ay)} = \sqrt{a^2xy} = a\sqrt{xy} = af(x, y)$$

Homogeneous of degree 1 or linearly homogenous.

(b)

$$f(x, y) = \left(x^2 - y^2\right)^{1/2}$$
$$f(ax, ay) = \left((ax)^2 - (ay)^2\right)^{1/2}$$
$$= \left(a^2x^2 - a^2y^2\right)^{1/2}$$
$$= \left(a^2\right)^{1/2}\left(x^2 - y^2\right)^{1/2} = af(x, y)$$

Homogeneous of degree 1.

(c) $f(x, y) = x^3 - xy + y^3$

$$f(ax, ay) = a^3x^3 - a^2xy + a^3y^3$$

Not homogenous.

(d) Homogeneous of degree 1.

(e) Homogeneous of degree 2.

(f) Homogeneous of degree 4.

2. Say we are given a production function $Q = f(K, L)$ that is homogenous of degree 1 or linearly homogenous.

Then dividing and multiplying by $K$ :

$$Q = K \cdot \frac{Q}{K} = K \cdot f\left(\frac{K}{K}, \frac{L}{K}\right) = K \cdot f\left(1, \frac{L}{K}\right) = K \cdot \psi\left(\frac{L}{K}\right)$$

Similarly, dividing and multiplying by $L$ :

$$Q = L \cdot \frac{Q}{L} = L \cdot f\left(\frac{K}{L}, \frac{L}{L}\right) = L \cdot f\left(\frac{K}{L}, 1\right) = L \cdot \phi\left(\frac{K}{L}\right)$$

6.

$$Q = AK^{\alpha}L^{\beta}$$

(a) and (b)

$$f(aK, aL) = A(aK)^{\alpha}(aL)^{\beta} = Aa^{(\alpha+\beta)}K^{\alpha}L^{\beta} = a^{\alpha+\beta}f(K, L)$$

When $\alpha + \beta > 1$, we have increasing returns to scale i.e. if we increase capital and labor by $a$-fold, output increases by more than $a$-fold. For eg. if we double $K$ and $L$, ie. $a = 2, Q$ increases by $2^{\alpha+\beta}$, which is more than double when $\alpha + \beta > 1$. Analogously, when $\alpha + \beta < 1$, we have decreasing returns to scale, and when $\alpha + \beta = 1$, we have constant returns to scale.

(c)

$$\frac{dQ}{dK} = \alpha AK^{\alpha-1}L^{\beta}$$
$$\frac{dQ}{dL} = \beta AK^{\alpha}L^{\beta-1}$$
$$\varepsilon_{Q,K} = \frac{dQ}{dK}\cdot\frac{K}{Q} = \frac{\alpha AK^{\alpha-1}L^{\beta}}{AK^{\alpha}L^{\beta}}\cdot K = \alpha$$
$$\varepsilon_{Q,L} = \frac{dQ}{dL}\cdot\frac{L}{Q} = \frac{\beta AK^{\alpha}L^{\beta-1}}{AK^{\alpha}L^{\beta}}\cdot L = \beta$$

7.

$$Q = AK^{a}L^{b}N^{c}$$

(a) $f(dk, dL, dN) = d^{a+b+c}f(k, L, N)$. Homogeneous of degree $a + b + c$.

(b) When $a+b+c = 1$, constant returns to scale. When $a+b+c > 0$, increasing returns to scale.

(c) Marginal product of factor $N$:

$$Q_N = \frac{dQ}{dN} = cAK^{a}L^{b}N^{c-1}$$

If $N$ is paid it's marginal product, total payment to factor $N$ is $N \cdot Q_N$. So it's share in the output is given by:

$$\frac{N\cdot Q_N}{Q} = N\cdot\frac{cAk^{a}L^{b}N^{c-1}}{AK^{a}L^{b}N^{c}} = c$$