Nonsingularity, Determinant, Matrix Inversion

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Determinant $|A|$ is a unique scalar associated with a square matrix A.

Determinant of a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Can be calculated as:

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

Find the determinant of A and B given below:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Is A nonsingular? What about B? Check if you get the same answer by reducing A and B to their echelon form and then finding the rank.

Determinant of a 3×3 matrix:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Find the determinant for:

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

Properties of Determinants

1. $|A| = |A'|$
2. Interchanging rows or columns will alter the sign but not the value
3. Multiplication of any one row (or one column) by a scalar k will change the value of the determinant k-fold
4. The addition (subtraction) of a multiple of any row (or column) to (from) another row (or column) will leave the determinant unaltered
5. If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Verify the above properties for a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The **minor** of the element a_{ij}, denoted by $|M_{ij}|$ is obtained by deleting the ith row and jth column of the matrix and taking the determinant of the resulting matrix.

Whereas, **cofactor** $|C_{ij}|$ is defined as:

$$|C_{ij}| = (-1)^{i+j}|M_{ij}|$$

Example.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor for the element a_{12}:

$$|M_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Cofactor for the element a_{12}:

$$|C_{12}| = (-1)^{(1+2)}\,|M_{12}| = -\,|M_{12}|$$

Find $|C_{31}|$, $|C_{32}|$, and $|C_{33}|$ for

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

Determinant for an $n \times n$ matrix is given by:

$$|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}| = \sum_{j=1}^{n} a_{ij}|C_{ij}|$$

The first expression corresponds to expanding with respect to the jth column, while the second expression is the expression for the determinant when expanding with respect to the ith row.

Find the determinant of $|A|$ by expanding with the third row.

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

To find the inverse of a nonsingular matrix A take the transpose of its cofactor matrix $C = [|C_{ij}|]$ to find the adjoint of A and divide it by the determinant of A.

$$A^{-1} = \frac{1}{|A|} adj A$$

Adjoint of a nonsingular $n \times n$ matrix

$$adj A = C' = \begin{bmatrix} |C_{11}| & |C_{21}| & \dots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \dots & |C_{n2}| \\ \vdots & \vdots & \dots & \vdots \\ |C_{1n}| & |C_{2n}| & \dots & |C_{nn}| \end{bmatrix}$$

Find the inverse of

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$