ECON 441

Introduction to Mathematical Economics

Div Bhagia

Lecture 4: Linear Algebra

Determinant of a $n \times n$ Matrix

A *minor* of the element a_{ij}, denoted by $|M_{ij}|$ is obtained by deleting the ith row and jth column.

Cofactor C_{ij} is defined as:

$$|C_{ij}| = (-1)^{i+j}|M_{ij}|$$

Then,

$$|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}| = \sum_{j=1}^{n} a_{ij}|C_{ij}|$$

Find the Determinant

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

Properties of Determinants

1. $|A| = |A'|$
2. Interchanging rows or columns will alter the sign but not the value
3. Multiplication of any one row (or one column) by a scalar k will change the value of the determinant k-fold
4. The addition (subtraction) of a multiple of any row (or column) to (from) another row (or column) will leave the determinant unaltered
5. If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Criteria for Nonsingularity

The following statements are equivalent:

- $|A| \neq 0$
- Rows (or equivalently columns) of A are independent
- A has full rank
- A is nonsingular
- A^{-1} exists
- A unique solution to $Ax = b$ ($x^* = A^{-1}b$) exists

Matrix Inversion

Adjoint of a nonsingular $n \times n$ matrix

$$adjA = C' = \begin{bmatrix} |C_{11}| & |C_{21}| & \dots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \dots & |C_{n2}| \\ \vdots & \vdots & \dots & \vdots \\ |C_{1n}| & |C_{2n}| & \dots & |C_{nn}| \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} AdjA$$

Find the Inverse

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

Find the Inverse

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

A Simple Economic Model

Two equations in two unknowns:

$$q + 2p = 100$$
$$q - 3p = 20$$

Can write this as:

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

Solution using Matrix Inversion

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

Cramer's Rule

More efficient way of solving a system of equations

The kth element of x can be solved by:

$$x_k^* = \frac{|A_k|}{|A|}$$

where A_k is a matrix formed by exchanging kth column of A by b.

Solution using Cramer's Rule

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

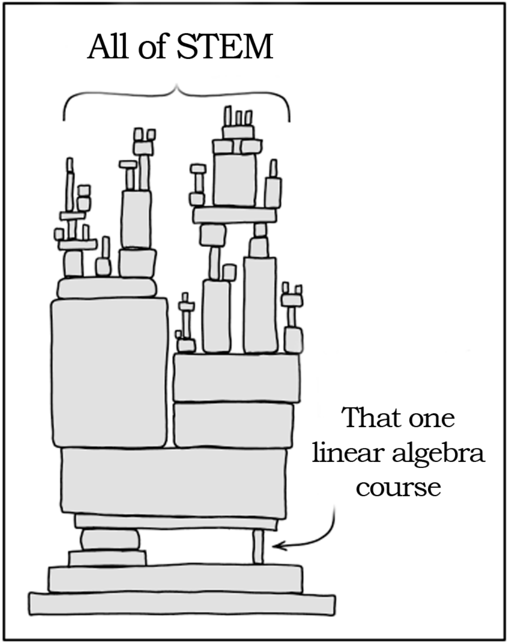
Homogeneous equation system

A homogeneous equation system is given by

$$Ax = 0$$

If A is nonsingular, $x^* = A^{-1}0 = 0$.

If A is singular there can be infinite number of solutions (this is true for any system of equations).



Coming up:

Applications of Matrix Algebra

Network Theory

A network of connections can be expressed as an adjacency matrix.

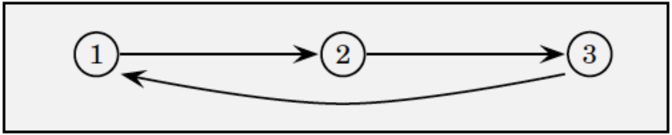
$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}$$

where

$$m_{ij} = \begin{cases} 1 & \text{if there is a direct link from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

Network Theory

Consider the following network:



$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrices M, M^2, M^3 give the nodes reachable in one, two and three steps from any initial node.

Network Theory

Consider the sum:

$$S_k = M + M^2 + M^3 + ... + M^k$$

The (i, j) element of S_k gives the number of paths of length k or less, from i to j.

For the previous example:

$$S_3 = M + M^2 + M^3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

So there is one way to go from any node to any other in three or fewer steps.

Uses of Network Theory

Network theory can be used to model

- Interconnectedness of financial institutions (to predict risk of banking collapses)
- Interconnectedness of the countries in world trade
- Predicting supply chain risk

Markov Chain

- A Markov Chain can model the transition between different states.
- Example: Employment (E) and Unemployment (U).
- Transition matrix:

$$P = \begin{pmatrix} P(E \to E) & P(U \to E) \\ P(E \to U) & P(U \to U) \end{pmatrix} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$$

- Say the initial state vector is:

$$\pi(0) = \begin{bmatrix} \pi_E(0) \\ \pi_U(0) \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

Transition Matrix

After one period, the state distribution is:

$$\pi(1) = \begin{bmatrix} \pi_E(1) \\ \pi_U(1) \end{bmatrix} = P\pi(0) = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.76 \\ 0.24 \end{bmatrix}$$

After t periods:

$$\pi(t) = P^t \pi(0)$$

Ordinary Least Squares

Linear model with k variables:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_k X_{ik} + \varepsilon_i$$

where $i = 1, ..., n$ denotes n observations.
Denote

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_{n\times 1}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}_{k\times 1}, \boldsymbol{X} = \begin{bmatrix} 1 & X_{11} & \ldots & X_{1k} \\ 1 & X_{12} & \ldots & X_{2k} \\ \vdots & \vdots & & \\ 1 & X_{1n} & \ldots & X_{nk} \end{bmatrix}_{n\times k}, \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n\times}$$

Then,

$$Y = X\beta + \varepsilon \qquad \text{OLS estimator: } \hat{\beta} = (X'X)^{-1}X'Y$$

Natural Language Processing (NLP)

- Bag of Words (BoW) model is a simple and widely used method in NLP
- Transform text into fixed-length vectors by counting how many times each word appears in a document
- Example:
  - Doc1: "the cat sat on the mat"
  - Doc2: "the dog sat on the log"
  - Vocabulary for these documents: $[the, cat, sat, on, mat, dog, log]$
  - Vector for Doc1: $[2, 1, 1, 1, 1, 0, 0]$
  - Vector for Doc2: $[2, 0, 1, 1, 0, 1, 1]$
- Calculate similarity between document vectors to classify documents into predefined classes

What's next?

- Quiz 2 next week will cover all of Linear Algebra
- Notes for reviewing Linear Algebra uploaded (not a substitute for the textbook for understanding concepts)
- We will move on to differential calculus next week

Homework Problems

Textbook reference: Sections 5.3-5.5

- Exercise 5.3: 1, 4, 5, 8
- Exercise 5.4: 2, 3, 4, 6, 7
- Exercise 5.5: 1, 2, 3 (a) (d)