# Homework 5 Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

*Exercise 6.2*

2. $y = 5x^2 - 4x$

   (a)
$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \frac{5(x+\Delta x)^2 - 4(x+\Delta x) - \left(5x^2 - 4x\right)}{\Delta x} \\ &= \frac{5\left(x^2 + \Delta x^2 + 2x\Delta x\right) - 4x - 4\Delta x - 5x^2 + 4x}{\Delta x} \\ &= \frac{5\Delta x^2 + 10x\Delta x - 4\Delta x}{\Delta x} = 5\Delta x + 10x - 4\end{aligned}$$

   (b)
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 5\Delta x + 10x - 4 = 10x - 4$$

   (c)
$$f'(2) = 10 \times 2 - 4 = 16$$
$$f'(3) = 10 \times 3 - 4 = 26$$

3. $y = 5x - 2$

   (a)
$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{5(\Delta x + x) - 2 - (5x - 2)}{\Delta x} \\ &= \frac{5\Delta x}{\Delta x} = 5\end{aligned}$$

   It is a constant function.

   (b) No it doesn't matter whether $\Delta x$ is small or large. $\frac{dy}{dx} = 5$

*Exercise 7.1*

3. (a) $f'(x) = 18$
   $f'(1) = 18$ and $f'(2) = 18$.

   (b) $f'(x) = 3cx^2$
   $f'(1) = 3c$ and $f'(2) = 12c$.

   (c) $f'(x) = 10x^{-3}$
   $f'(1) = 10$ and $f'(2) = \frac{10}{8} = \frac{5}{4}$

   (d) $f'(x) = x^{1/3} = \sqrt[3]{x}$
   $f'(1) = 1$ and $f'(2) = \sqrt[3]{2}$

   (e) $f'(w) = 2w^{-2/3}$
   $f'(1) = 2$ and $f'(2) = 2 \cdot 2^{-2/3} = 2^{1/3}$

   (f) $f'(w) = \frac{1}{2}w^{-7/6}$
   $f'(1) = \frac{1}{2}$ and $f'(2) = \frac{1}{2}\left(2^{-7/6}\right) = 2^{-1} \cdot 2^{-7/6}$

*Exercise 7.2*

3. (d) $\underbrace{(ax-b)}_{f(x)}\underbrace{\left(cx^2\right)}_{g(x)}$

   $f'(x) = a$
   $g'(x) = 2cx$

$$\begin{aligned}\frac{d}{dx}f(x)g(x) &= f'(x)g(x) + f(x)g'(x) \\ &= acx^2 + (ax-b)2cx \\ &= acx^2 + 2acx^2 - 2bcx \\ &= 3acx^2 - 2bcx\end{aligned}$$

(e) $\underbrace{(2-3x)(1+x)}_{f(x)}\underbrace{(x+2)}_{g(x)}$

$g'(x) = 1$

$f(x) = \underbrace{(2-3x)}_{h(x)}\underbrace{1+x)}_{p(x)}$

$$\begin{aligned} f'(x) &= h'(x)p(x) + h(x)p'(x) \\ &= -3(1+x) + (2-3x) \\ &= -(6x+1) \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}f(x)g(x) &= f'(x)g(x) + f(x)g'(x) \\ &= -(6x+1)(x+2) + (2-3x)(1+x) \\ &= -6x^2 - x - 12x - 2 + 2 - 3x + 2x - 3x^2 \\ &= -9x^2 - 14x \\ &= -x(9x+14) \end{aligned}$$

7. (a)

$$\frac{x^2+3}{x}$$

$$\begin{aligned} f(x) &= x^2 + 3 \quad \to f'(x) = 2x \\ g(x) &= x \qquad\quad\; \to g'(x) = 1 \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{(2x)x - \left(x^2+3\right)\cdot 1}{x^2} \\ &= \frac{2x^2 - x^2 - 3}{x^2} = \frac{x^2-3}{x^2} \end{aligned}$$

(b)

$$f(x) = \frac{x+9}{x}$$

$$f'(x) = \frac{1 \cdot x - 1 \cdot (x+9)}{x^2} = \frac{-9}{x^2}$$

(c)

$$f(x) = \frac{6x}{x+5}$$
$$f'(x) = \frac{6 \cdot (x+5) - 1.6x}{(x+5)^2} = \frac{30}{(x+5)^2}$$

(d)

$$f(x) = \frac{ax^2+b}{cx+d}$$

$$\begin{aligned} f'(x) &= \frac{2ax(cx+d) - \left(ax^2+b\right)c}{(cx+d)^2} \\ &= \frac{2acx^2 + 2adx - cax^2 - bc}{(cx+d)^2} \\ &= \frac{acx^2 + 2adx - bc}{(cx+d)^2} \end{aligned}$$

8. $f(x) = ax + b$

(a) $f'(x) = a$

(b) $\frac{d}{dx}ax^2 + bx = 2ax + b$

(c)

$$\frac{d}{dx}\frac{1}{ax+b} = \frac{0(ax+b) - (a) \cdot 1}{(ax+b)^2} = \frac{-a}{(ax+b)^2}$$

(d)

$$\frac{d}{dx}\frac{ax+b}{x} = \frac{ax - (ax+b) \cdot 1}{x^2} = \frac{-b}{x^2}$$

*Exercise 7.3*

1.

$$y = u^3 + 2u$$
$$u = 5 - x^2$$

$$\frac{dy}{du} = 3u^2 + 2$$
$$\frac{du}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{dy}{dy} \cdot \frac{dy}{dx} = \left(3y^2 + 2\right)(-2x)$$
$$= \left(3\left(5 - x^2\right)^2 + 2\right)(-2x)$$

2.

$$w = ay^2$$
$$y = bx^2 + cx$$

$$\frac{dw}{dx} = \frac{dw}{dy} \cdot \frac{dy}{dx}$$
$$= 2ay(2bx + c)$$
$$= 2a\left(bx^2 + cx\right)(2bx + c)$$

3. (a) $y = \left(3x^2 - 13\right)^3$

Denote, $z = 3x^2 - 13$

Then, $y = z^3$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 3z^2(6x) = 18x\left(3x^2 - 13\right)^2$$

(b)

$$y = \left(7x^3 - 5\right)^9$$
$$z = 7x^3 - 5 \rightarrow y = z^9$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 9z^8\left(21x^2\right) = 189x^2\left(7x^3 - 5\right)^8$$

(c)

$$y = (ax + b)^5$$
$$z = ax + b \rightarrow y = z^5$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 5z^4(a) = 5a(ax + b)^4$$

4.

$$y = (16x + 3)^{-2}$$
$$z = 16x + 3$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = -32(16x + 3)^{-3}$$

Using the quotient rule:

$$y = \frac{1}{(16x + 3)^2}$$

$$\frac{dy}{dx} = \frac{0 \cdot (16x + 3)^2 - 32(16x + 3)}{(16x + 3)^4}$$
$$= \frac{-32}{(16x + 3)^{-3}}$$

5.

$$y = 7x + 21 \rightarrow \frac{dy}{dx} = 7$$

$$x = \frac{y-21}{7} \rightarrow \frac{dx}{dy} = \frac{1}{7}$$

6. (a)

$$\frac{dy}{dx} = -6x^5 < 0$$

For $x > 0$, this function is strictly decreasing as its derivative is negative.

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{-1}{6x^5}$$

(b)

$$y = 4x^5 + x^3 + 3x$$

$$\frac{dy}{dx} = 20x^4 + 3x^2 + 3 > 0$$

This function is strictly increasing as $\frac{dy}{dx} > 0$ for all $x$.

$$\frac{dx}{dy} = \frac{1}{20x^4 + 3x^2 + 3}$$