ECON 441

Introduction to Mathematical Economics

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Lecture 8: Unconstrained Single Variable Optimization,
Concave & Convex Functions

Optimization

How many hours to work each week?

Utility from consumption (C) and leisure (L):

$$U(C, L)$$

Leisure is the difference between total hours (T) and hours worked (H)

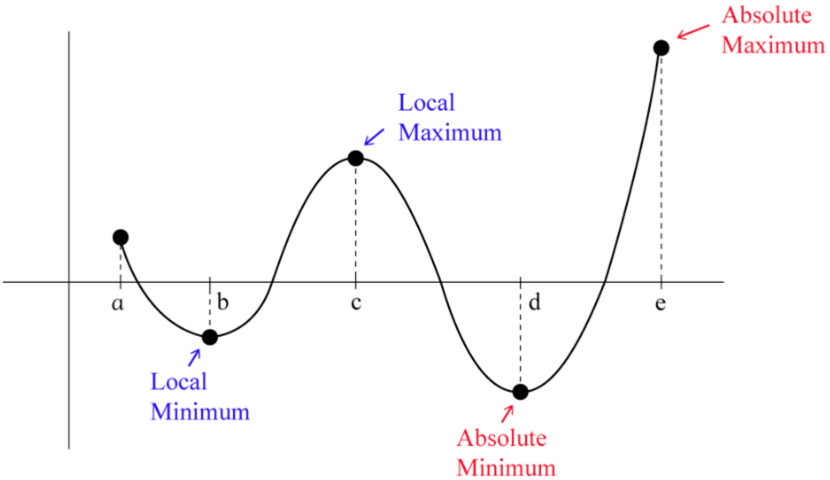
$$L = T - H$$

Constraint:

$$wH = pC$$

where w and p denote wages and price, respectively.

Global vs Local Extrema



Critical Points

Limit ourselves to functions that are *continuously differentiable* i.e. f is continuous and has a continuous derivative.

$$y = f(x)$$

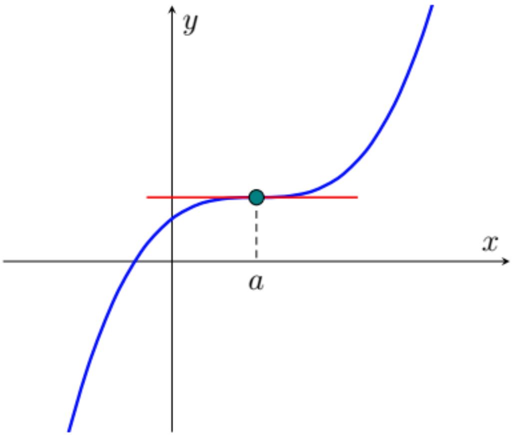
A *necessary* condition for x_0 to be an extremum

$$f'(x_0) = 0$$

x_0 is called a critical/stationary point.

Why not sufficient?

Inflection Point

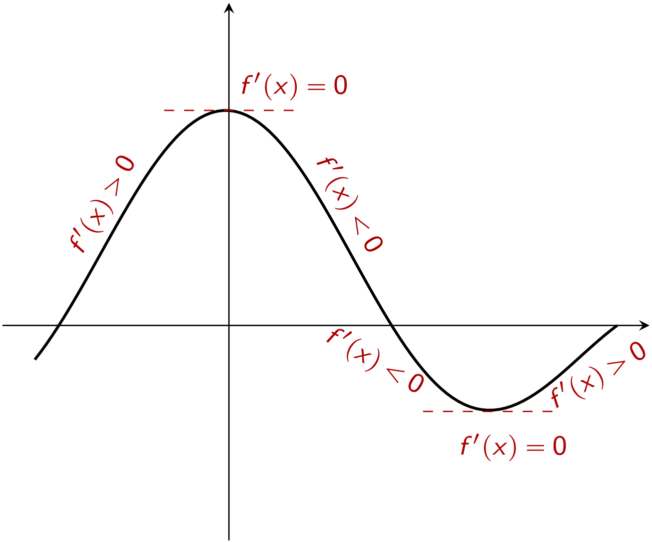


First-Derivative Test

Suppose that $f'(x_0) = 0$. Then $f(x_0)$ is a

1. maximum if $f'(x)$ goes from $+$ to $-$ in the immediate neighborhood of x_0
2. minimum if $f'(x)$ goes from $-$ to $+$ in the immediate neighborhood of x_0
3. not an extreme point if $f'(x)$ has the same sign in its immediate neighborhood

First-Derivative Test



Example

Let's find the extrema for the following function:

$$f(x) = x^2 - 24x + 36$$

Example

Total cost:

$$TC = C(Q)$$

Marginal cost:

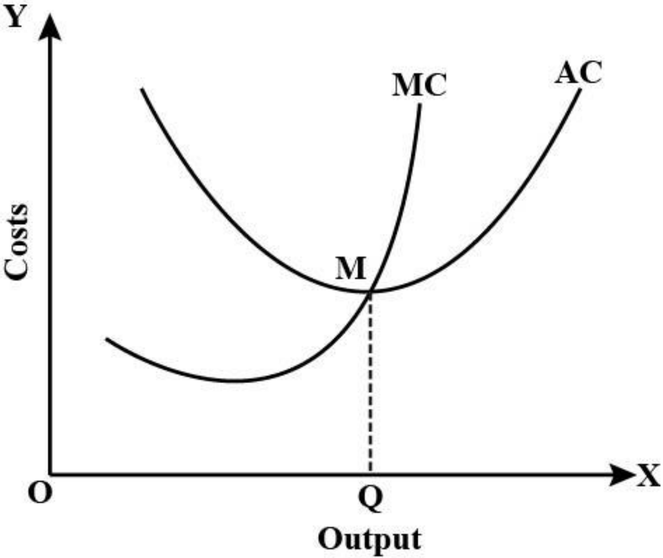
$$MC = C'(Q)$$

Average cost:

$$AC = \frac{C(Q)}{Q}$$

At what quantity is AC the lowest?

Average and Marginal Cost



Second and Higher Derivatives

The derivative of $f'(x)$ is called the second derivative and is denoted by:

$$f''(x) = \frac{d^2y}{dx^2}$$

Similarly, we can obtain other higher-order derivatives:

$$f^3(x), f^{(4)}(x), \cdots, f^{(n)}(x)$$

or

$$\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \dots, \frac{d^ny}{dx^n}$$

Second Derivative

With an infinitesimal increase in x from x_0

- *Value* of the function increases if $f'(x_0) > 0$
- *Value* of the function decreases if $f'(x_0) < 0$

- *Slope* of the function increases if $f''(x_0) > 0$
- *Slope* of the function decreases if $f''(x_0) < 0$

Second Derivative Test

If $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$ will be

1. a maximum if $f''(x_0) < 0$
2. a minimum if $f''(x_0) > 0$

More convenient to use than the first-derivative test.

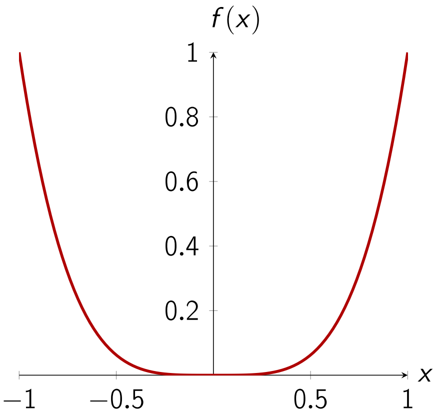
Necessary vs Sufficient Conds.

Condition	Maximum	Minimum
First-order necessary	$f'(x) = 0$	$f'(x) = 0$
Second-order necessary †	$f''(x) \leqslant 0$	$f''(x) \geqslant 0$
Second-order sufficient †	$f''(x) < 0$	$f''(x) > 0$

† Applicable only after the first-order necessary condition has been satisfied.

Necessary vs Sufficient Conds.

$$y = x^4$$



Concave and Convex Functions

- Concave function: $f''(x) \leqslant 0$ for all x
- Convex function: $f''(x) \geqslant 0$ for all x

- Strictly concave function: $f''(x) < 0$ for all x
- Strictly convex function: $f''(x) > 0$ for all x

Concave and Convex Functions

f is concave if:

$$f(\lambda x_1 + (1-\lambda)x_2) \geqslant \lambda f(x_1) + (1-\lambda)f(x_2)$$

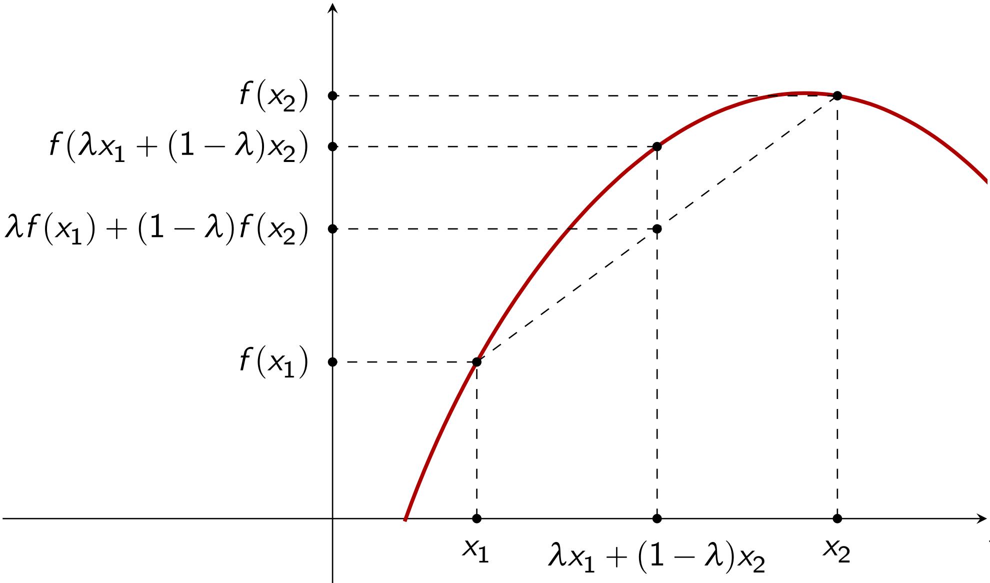
f is covex if:

$$f(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda f(x_1) + (1-\lambda)f(x_2)$$

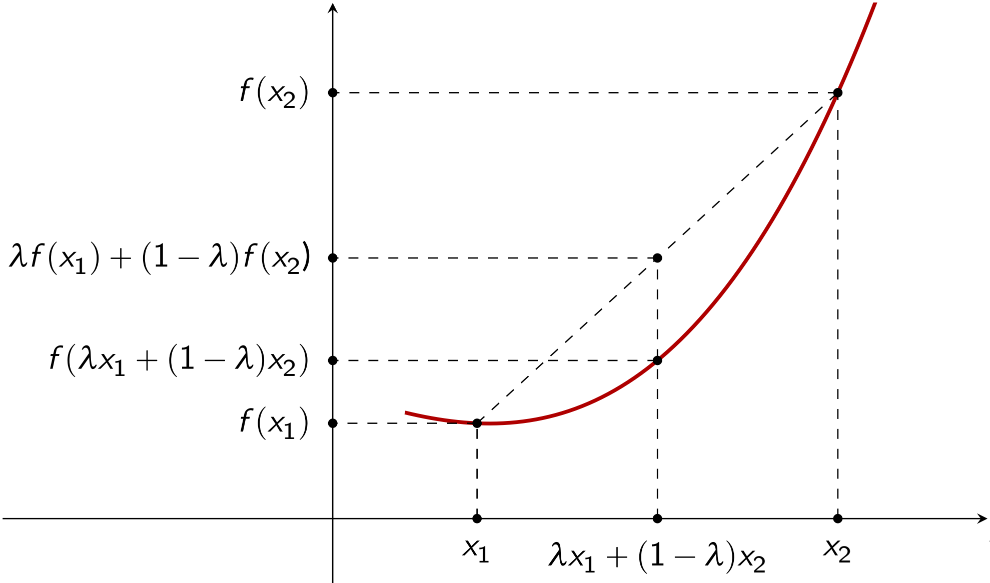
where $\lambda \in (0,1)$.

For strict concavity/convexity replace with strict inequalities.

Concave Function



Convex Function



Attitudes toward Risk

Consider the following game: flip a coin, collect \$0 if tails, collect \$20 if heads.

How much would you pay to play this game?

Concave and Convex Functions

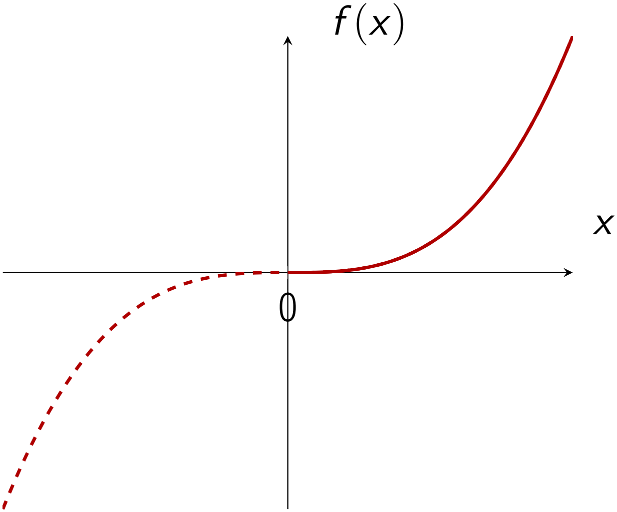
The domain of f is all real numbers

$$f(x) = x^3$$

Is f a convex, strictly convex, concave, or strictly concave function?

What if the domain of f is all nonnegative real numbers?

$f(x) = x^3$



Properties of Concave and Covex Functions

1. If $f(x)$ is a linear function, then it is a concave function as well as a convex function, but not strictly so.

2. If $f(x)$ is a (strictly) concave function, then $-f(x)$ is a (strictly) convex function, and vice versa.

3. If $f(x)$ and $g(x)$ are both concave (convex) functions, then $f(x) + g(x)$ is a concave (convex) function. Further, in addition, either one or both of them are strictly concave (strictly convex), then $f(x) + g(x)$ is strictly concave (convex).

Global Optimizers

- If a function is concave, any critical point will give us a global maximum.
- If a function is strictly concave, any critical point will give us the *unique* global maximum.
- If a function is convex, any critical point will give us a global minimum.
- If a function is strictly convex, any critical point will give us the *unique* global minimum.

Concave and Convex Functions

Say, $f(x)$ is a strictly concave function and

$$f'(2) = 0$$

Is $f(2)$ the local or global maximum? Is it the unique maximum?

References and Homework

- Covered today: Sections 9.1, 9.2, 9.3, 9.4
- Homework problems:
  - Exercise 9.2: 1 (c), 2 (a), 3, 4
  - Exercise 9.3: 2, 3, 4, 5
  - Exercise 9.4: 1 (b) (d), 2, 3, 5