Homework 8 Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Exercise 9.2

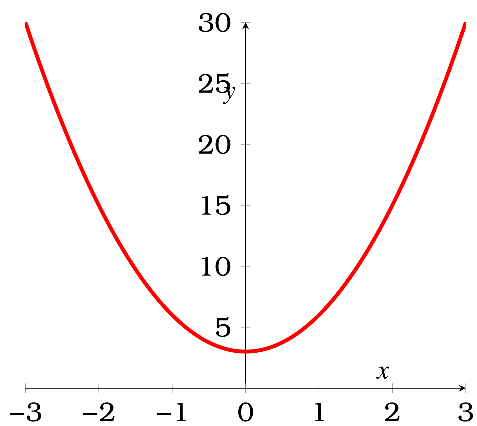
1. (c) $y = 3x^2 + 3$

$$\frac{dy}{dx} = 6x = 0 \rightarrow x^* = 0$$

In the immediate neighborhood of 0, for $x < 0$, $\frac{dy}{dx} < 0$, while for $x > 0$, $\frac{dy}{dx} > 0$. This implies that at 0, the slope of the function changes sign from negative to positive i.e. the function was decreasing on the left of 0 but is increasing on the right. So it must be that the function has a relative minimum ($f(0) = 3$) at $x = 0$. We can also confirm this by looking at the 2nd derivative:

$$\frac{d^2y}{dx^2} = 6 > 0$$

The graph of this function is given below:



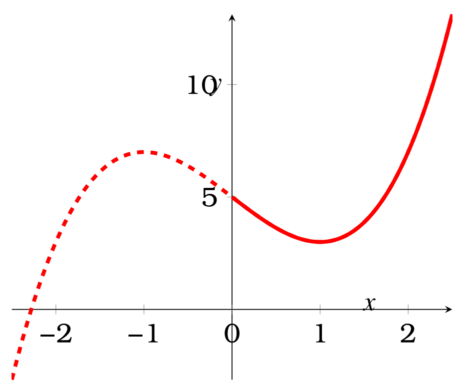
2. (a) $y = x^3 - 3x + 5$

$$\frac{dy}{dx} = 3x^2 - 3 = 0 \rightarrow x^* = \pm\sqrt{1}$$

So, we have two critical values $x_1^* = 1$ and $x_2^* = -1$.

The derivative of the function, $3\left(x^2 - 1\right)$, is negative on the immediate left of 1 (e.g. 0.9) and is positive on the immediate right of 1 (e.g. 1.1). While it is positive on the immediate left of -1 (e.g. -1.1) but negative on the right (e.g. -0.9). So the function should have a relative minimum at 1 and a relative maximum at -1. However, the domain of this function is limited to positive real numbers, in which case -1 is not permissible. So we only have a relative minimum $f(1) = 3$.

The graph (solid line) of this function is given below:



We could have also reached the above conclusion from the second derivative test.

$$\frac{d^2y}{dx^2} = 6x$$

$\frac{d^2y}{dx^2} > 1$ when $x = 1 \rightarrow 1$ relative minimum at 1

$\frac{d^2y}{dx^2} < 0$ when $x = -1 \rightarrow -1$ relative maximum at -1

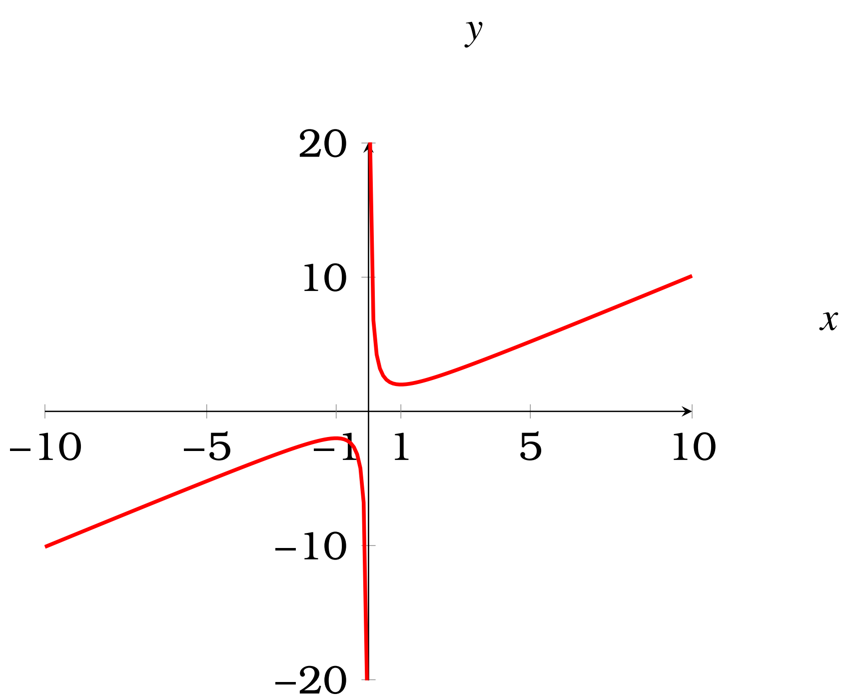
3. $f(x) = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2} = 0 \rightarrow x^* = \pm 1$$

The derivative of the function, $(x^2 - 1)/x^2$, is negative on the immediate left of 1 (e.g. 0.9) and is positive on the immediate right of 1 (e.g. 1.1). While it is positive on the immediate left of -1 (e.g. -1.1) but negative on the right (e.g. -0.9). So the function should have a relative minimum at 1 and a relative maximum at -1.

$$f(1) = 2$$
$$f(-1) = -2$$

Here, the relative maximum $f(-1) = 0$ is lower than the relative minimum $f(1) = 2$. However, it is still correct as these are just *relative* extrema. The graph for this function clarifies this notion.



4. $T = \phi(x)$

   (a) $M = \phi'(x)$

   (b) $A = \phi(x)/x$

   (c) Critical point:

$$A' = \frac{\phi'(x)x - \phi(x)}{x^2} = 0 \rightarrow \phi'(x^*) = \frac{\phi(x^*)}{x^*}$$

   (d) Elasticity of T:

$$\varepsilon = \frac{\phi'(x)x}{\phi(x)} = \frac{M}{A}$$

   When $M = A \rightarrow \varepsilon = 1$

Exercise 9.3

2. (a) $f(x) = 9x^2 - 4x + 8$

$$f'(x) = 18x - 4$$
$$f''(x) = 18 > 0$$

   The function is strictly convex.

   (b) $w = -3x^2 + 39$

$$\frac{dw}{dx} = -6x$$
$$\frac{d^2w}{dx^2} = -6 < 0$$

   The function is strictly concave.

   (c) $u = 9 - 2x^2$

$$f'(x) = -4x$$
$$f''(x) = -4 < 0$$

   The function is strictly concave.

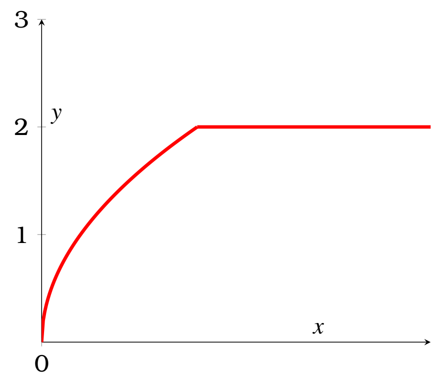
(d) $v = 8 - 5x + x^2$

$$\frac{dv}{dx} = -5 + 2x$$

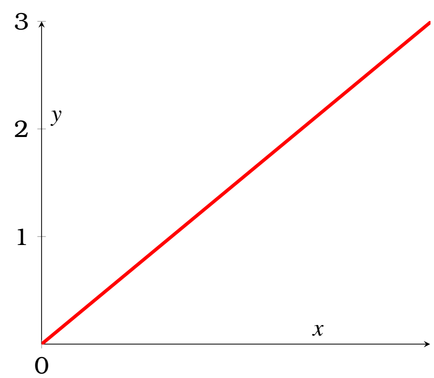
$$\frac{d^2v}{dx^2} = 2 > 0$$

The function is strictly convex.

3. (a) Concave but not strictly concave



(b) Concave and convex



4. We are given the following function:

$$y = a - \frac{b}{c+x} \quad (a, b, c > 0, x \geqslant 0)$$

(a)

$$\frac{dy}{dx} = \frac{b}{(c+x)^2} > 0$$

$$\frac{d^2y}{dx^2} = \frac{-b}{(c+x)^4} \cdot 2(c+x)$$

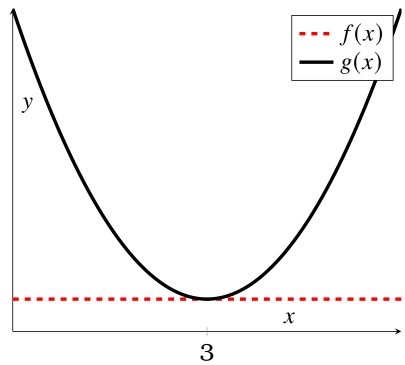
$$= \frac{-2b}{(c+x)^3} < 0$$

(b) When $x = 0$,

$$y = a - \frac{b}{c}$$

(c) As $x \to \infty, y \to a$

We should restrict $ac > b$ to ensure consumption is positive. We should also make sure consumption (y) does not increase more than one-to-one with income (x), such that $dy/dx < 1$, so $b < c^2$.

5. 

$f(x)$ has infinitely many stationary points, while $g(x)$ has one stationary point 3.

Exercise 9.4

1. (b)

$$\begin{aligned} f(x) &= x^3 + 6x^2 + 9 \\ f'(x) &= 3x^2 + 12x \\ &= 3x(x+4) \to x^* = 0, -4 \\ f''(x) &= 6x + 12 \\ f''(0) &= 12 > 0 \to f(0) = 9 \text{ is a local min} \\ f''(-4) &= -24 + 12 = -12 \to f(-4) = 41 \text{ is a local max} \end{aligned}$$

2.

$$A = xy$$
$$2x + y = 64 \rightarrow y = 64 - 2x$$
$$A = x(64 - 2x) = 64x - 2x^2$$
$$\frac{dA}{dx} = 64 - 4x \rightarrow x^* = 16$$

To see if it is indeed the maximum:

$$\frac{d^2A}{dx^2} = -4 < 0$$

3. (a) Yes

(b)

$$\begin{aligned} R = PQ &= (100 - Q)Q \\ &= 100Q - Q^2 \end{aligned}$$

(c) $\pi = R - C$

$$= 100Q - Q^2 - \frac{1}{3}Q^3 + 7Q^2 - 111Q - 50$$
$$= -\frac{1}{3}Q^3 + 6Q^2 - 11Q - 50$$

(d) $\frac{d\pi}{dQ} = -Q^2 + 12Q - 11 = 0$

$$Q^2 - 12Q + 11 = 0$$

$$Q^2 - 11Q - Q + 11 = 0$$
$$Q(Q - 11) - 1(Q - 11) = 0$$
$$(Q - 1)(Q - 11) = 0 \rightarrow Q^* = 1 \text{ and } 11$$

$$\frac{d^2\pi}{dQ^2} = -2Q + 12$$

At $Q = 1, -2 + 12 = 10 > 0$

At $Q = 11, -22 + 12 = -10 < 0$, so profit maximizing $Q = 11$.

(e)

$$\begin{aligned}\pi =& -\frac{1}{3}Q^3 + 6Q^2 - 11Q - 50\\ =& Q\left[Q\left[-\frac{1}{3}Q + 6\right] - 11\right] - 50\end{aligned}$$

$$\begin{aligned}\text{Max}\,\pi =& 11\left[11\left(6 - \frac{11}{3}\right) - 11\right] - 50\\ & 11\left(11 \times \left(\frac{7}{3} - 1\right)\right] - 50 = 40.75\end{aligned}$$

5. Profit function:

$$\pi(Q) = hQ^2 + jQ + k$$

(a) $\pi(0) = k < 0$

(b) $\pi'(Q) = 2hQ + j, \quad \pi''(Q) = 2h < 0 \rightarrow h < 0$

(c) Critical point: $\pi'(Q) = 2hQ + j = 0 \rightarrow Q^* = -j/2h$. Since we assumed $h < 0$, assuming $j > 0$ ensures $Q^* > 0$.